※ 多元回归中OLS估计量的性质推导.

(1) $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + u.$

OLS: $\hat{\beta}_1 = \dfrac{\sum_{i=1}^{n} \hat{r}_{i1} y_i}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$ (3.22)
for x_1

where $\hat{r}_{i1} = x_{i1} - \hat{\delta}_0 - \hat{\delta}_2 x_{i2} - \cdots - \hat{\delta}_k x_{ik}$

(该部分证明见p3)

(2) 代入 $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots \beta_k x_{ik} + u_i$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} \hat{r}_{i1}(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + u_i)}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$$

$$= \frac{\beta_0 \overbrace{\sum_{i=1}^{n} \hat{r}_{i1}}^{=0} + \beta_1 \overbrace{\sum_{i=1}^{n} \hat{r}_{i1} x_{i1}}^{=\sum_{i=1}^{n} \hat{r}_{i1}^2} + \cdots + \overbrace{\beta_k \sum_{i=1}^{n} \hat{r}_{i1} x_{ik}}^{=0} + \sum_{i=1}^{n} \hat{r}_{i1} u_i}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$$

by: $\sum_{i=1}^{n} \hat{r}_{i1} = 0$, $\sum_{i=1}^{n} x_{ij} \hat{r}_{i1} = 0 \ \forall j \geq 2$

$\sum_{i=1}^{n} x_{i1} \hat{r}_{i1} = \sum_{i=1}^{n} \hat{r}_{i1}^2$ (可由前两个条件推出).

$$= \beta_1 + \frac{\sum_{i=1}^{n} \hat{r}_{i1} u_i}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$$

① $\equiv w_{i1}$ 为 $\hat{r}_{i1}$ 的函数

其中：$\hat{r}_{i1}$ 为 x_{ij} 的函数

故：w_{i1} 非随机，只有 u_i 随机

② $u_i \sim N(0, \sigma^2)$，i.i.d. 独立同分布.

$\Rightarrow \beta_1 + w_{11} u_1 + w_{21} u_2 + \cdots + w_{n1} u_n$ 是 u_i 的线性组合.

以 x 为条件，则 x 可视为常数，则正态的线性组合仍为正态。

(Conditional on x)

正态性得证。

③ 求 $\hat{\beta}_1$ 的期望（以 x 为条件）：

$$E(\hat{\beta}_1) = E(\beta_1 + \sum_{i=1}^{n} w_{i1} u_i) = \beta_1 + \sum_{i=1}^{n} w_{i1} \underbrace{E(u_i)}_{=0}$$

$$= \beta_1$$

无偏性得证。

补充：(1)的证明

总体模型：$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + u$

① 根据 Frisch-Waugh 定理，$\hat{\beta}_1$ 可通过以下两步回归获得：

i. x_1 对其它解释变量回归，得到残差 $\hat{r}_{i1}$：

$$\hat{r}_{i1} = x_{i1} - \hat{\delta}_0 - \hat{\delta}_2 x_{i2} - \cdots - \hat{\delta}_k x_{ik}$$

ii. y 对 $\hat{r}_{i1}$ 回归，得到 $\hat{\beta}_1$：

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (\hat{r}_{i1} - \bar{\hat{r}}_1)(y_i - \bar{y})}{\sum_{i=1}^{n} (\hat{r}_{i1} - \bar{\hat{r}}_1)^2}$$ (简单回归公式)

其中，$\bar{\hat{r}}_1 = \frac{1}{n}\sum_{i=1}^{n} \hat{r}_{i1}$ 为 $\hat{r}_{i1}$ 的样本均值。

② 由简单回归的性质可得 $\bar{\hat{r}}_1 = 0$，则 $\hat{\beta}_1$ 可简化为：

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \hat{r}_{i1}(y_i - \bar{y})}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$$

③ 将分子式展开为：

$$\sum_{i=1}^{n} \hat{r}_{i1}(y_i - \bar{y}) = \hat{r}_{11}(y_1 - \bar{y}) + \hat{r}_{21}(y_2 - \bar{y}) + \cdots + \hat{r}_{n1}(y_n - \bar{y})$$

$$= \hat{r}_{11} y_1 + \hat{r}_{21} y_2 + \cdots + \hat{r}_{n1} y_n$$

$$+ \bar{y} \underbrace{(\hat{r}_{11} + \hat{r}_{21} + \cdots + \hat{r}_{n1})}_{= n\bar{\hat{r}}_1 = 0}$$

$$= \sum_{i=1}^{n} \hat{r}_{i1} y_i$$

故 $\hat{\beta}_1$ 可简化为:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \hat{r}_{i1} y_i}{\sum_{i=1}^{n} \hat{r}_{i1}^2}$$

证毕。